Calculus AB

Derivative Formulas

Derivative Notation:

For a function f(x), *the derivative would be* $f'(x)$

Leibniz's Notation:

For the derivative of y *in terms of* x, *we write* $\frac{dy}{dx}$

For the second derivative using Leibniz's Notation: $\frac{d^2y}{dx^2}$

Product Rule:

$$y = f(x)g(x)$$
$$\frac{dy}{dx} = f'(x)g(x) + g'(x)f(x)$$

$$y = x^2 \sin x$$
$$\frac{dy}{dx} = 2x \sin x + \cos x(x^2)$$
$$\frac{dy}{dx} = 2x \sin x + x^2 \cos x$$

Quotient Rule:

$$y = \frac{f(x)}{g(x)}$$
$$\frac{dy}{dx} = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

$$y = \frac{\sin x}{x^3}$$
$$\frac{dy}{dx} = \frac{\cos x(x^3) - 3x^2 \sin x}{x^6}$$
$$\frac{dy}{dx} = \frac{x \cos x - 3\sin x}{x^4}$$

Chain Rule:

$$y = (f(x))^n$$
$$\frac{dy}{dx} = n(f(x))^{n-1}(f'(x))$$

$$y = (x^2 + 1)^3$$
$$\frac{dy}{dx} = 3(x^2 + 1)^2 \cdot 2x$$
$$\frac{dy}{dx} = 6x(x^2 + 1)^2$$

Natural Log

$$y = \ln(f(x))$$
$$\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$$

$$y = \ln(x^2 + 1)$$
$$\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x$$
$$\frac{dy}{dx} = \frac{2x}{x^2 + 1}$$

Power Rule:

$$y = x^a$$
$$\frac{dy}{dx} = ax^{a-1}$$

$$y = 2x^5$$
$$\frac{dy}{dx} = 10x^4$$

Constant with a Variable Power:

$$y = a^{f(x)}$$
$$\frac{dy}{dx} = a^{f(x)} \cdot \ln a \cdot f'(x)$$

$$y = 2^x$$
$$\frac{dy}{dx} = 2^x \ln \cdot 1$$

$$y = 3^{x^2}$$
$$\frac{dy}{dx} = 3^{x^2} \cdot \ln 3 \cdot 2x$$

Variable with a Variable power

$y = f(x)^{g(x)}$ *Take* ln *of both sides!*

$$y = x^{\sin x}$$
$$\ln y = \ln x^{\sin x}$$
$$\ln y = \sin x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{1}{x}\sin x$$
$$\frac{dy}{dx} = x^{\sin x}[\cos x \ln x + \frac{1}{x}\sin x]$$

Implicit Differentiation:

Is done when the equation has mixed variables:

$$x^2 + x^2y^3 + y^4 = 5$$

$$derivative \Rightarrow 2x + [2xy^3 + 3y^2 \frac{dy}{dx} x^2] + 4y^3 \frac{dy}{dx} = 0$$

$$\Rightarrow 3y^2x^2 \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = -2x - 2xy^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x - 2xy^3}{3y^2x^2 + 4y^3}$$

Trigonometric Functions:

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

Inverse Trigonometric Functions:

$$y = \arcsin x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \cdot 1$$

$$y = \arcsin x^4$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^8}} \cdot 4x^3$$

$$y = \arctan x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \cdot 1$$

$$y = \arctan x$$

$$\frac{dy}{dx} = \frac{1}{1+x^6} \cdot 3x^2$$

Integral Formulas

Basic Integral

$$\int 5\,dx$$
$$= 5x + C$$

,where C *is an arbitrary constant*

$$\int \pi$$
$$= \pi x + C$$

Variable with a Constant Power

$$\int x^a dx$$
$$= \frac{x^{a+1}}{a+1} + C$$

$$\int x^3 dx$$
$$= \frac{x^4}{4} + C$$

Constant with a Variable Power

$$\int a^x dx$$
$$= \frac{a^x}{\ln a} + C$$

$$\int 5^x dx$$
$$= \frac{5^x}{\ln 5} + C$$

$$\int 3^{2x} dx$$
$$= \frac{3^{2x}}{2\ln 3} + C$$

Fractions

$$\int \frac{1}{x^4} dx$$
$$\int x^{-4} dx$$
$$= -\frac{x^{-3}}{3}$$

-unless-

if the top is the derivative of the bottom

$$\int \frac{1}{x}\,dx$$
$$= \ln|x| + C$$

$$\int \frac{x^3}{x^4+1}\,dx$$
$$= \frac{1}{4}\ln|x^4+1| + C$$

Substitution

When integrating a product in which the terms are somehow related, we usually let u= *the part in the parenthesis, the part under the radical, the denominator, the exponent, or the angle of the trigonometric function*

$$\int x\sqrt{x^2+1}\cdot dx; \quad u = x^2+1$$
$$du = 2x\cdot dx$$
$$= \frac{1}{2}\int 2x(x^2+1)^{1/2}\cdot dx$$
$$= \frac{1}{2}\int u^{1/2}du$$
$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}+C$$
$$= \frac{1}{3}(x^2+1)^{3/2}+C$$

$$\int \cos 2x\ dx; \quad u = 2x$$
$$du = 2\cdot dx$$
$$= \frac{1}{2}\int 2\cos 2x\cdot dx$$
$$= \frac{1}{2}\int \cos u\ du$$
$$= \frac{1}{2}\sin u + C$$
$$= \frac{1}{2}\sin 2x + C$$

Integration by Parts

When taking an integral of a product, substitute for u *the term whose derivative would eventually reach 0 and the other term for* dv.

The general form: $uv - \int vdu$ (pronounced "of dove")

Example:

$$\int x\cdot e^x\ dx$$
$$u = x \quad dv = e^x dx$$
$$du = 1\,dx \quad v = e^x$$

$$= x\cdot e^x - \int e^x dx$$
$$= xe^x - e^x + C$$

Example 2:

$$\int x^2 \cos x$$

$$u = x^2 \quad dv = \cos x \, dx$$

$$du = 2x\,dx \quad v = \sin x$$

$$x^2 \sin x - \int 2x \sin x$$

$$u = 2x \quad dv = \sin x \, dx$$

$$du = 2\,dx \quad v = -\cos x$$

$$x^2 \sin x - [-2x\cos x - \int -2\cos x$$

$$\Rightarrow x^2 \sin x + 2x\cos x - 2\sin x + C$$

Inverse Trig Functions

Formulas:

$$\int \frac{1}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C \qquad \int \frac{1}{a^2 + x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

Examples:

$$\int \frac{1}{\sqrt{9 - x^2}}; \quad a = 3; v = x \qquad \int \frac{1}{16 + x^2}; \quad a = 4; v = x$$

$$= \arcsin\frac{x}{3} + C \qquad = \frac{1}{4}\arctan\frac{x}{4} + C$$

More examples:

$$\int \frac{1}{\sqrt{4 - 9x^2}}; \quad a = 2; v = 3x$$

$$\frac{1}{3}\int \frac{3}{\sqrt{4 - 9x^2}}$$

$$= \frac{1}{3}\arcsin\frac{3x}{2} + C$$

$$\int \frac{1}{9x^2 + 16} \quad a = 4; v = 3x$$

$$\frac{1}{3}\int \frac{3}{9x^2 + 16}$$

$$= \frac{1}{3} \bullet \frac{1}{4}\arctan\frac{3x}{4} + C$$

$$= \frac{1}{12}\arctan\frac{3x}{4} + C$$

Trig Functions

$$\int \sin x\,dx = -\cos x + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \cos 2x\,dx = \frac{\sin 2x}{2} + C \qquad \int \sec x \tan x = \sec x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

Properties of Logarithms

Form

logarithmic form <=> exponential form

y = $\log_a x$ <=> $a^y = x$

Log properties

y = $\log x^3$ => y = 3 $\log x$

$\log x + \log y = \log xy$

$\log x - \log y = \log (x/y)$

Change of Base Law

This is a useful formula to know.

$$y = \log_a x \quad \Rightarrow \quad \frac{\log x}{\log a} \quad -or- \quad \frac{\ln x}{\ln a}$$

Properties of Derivatives

1st Derivative shows: maximum and minimum values, increasing and decreasing intervals, slope of the tangent line to the curve, and velocity

2nd Derivative shows: inflection points, concavity, and acceleration

- *Example on the next page* -

Example:

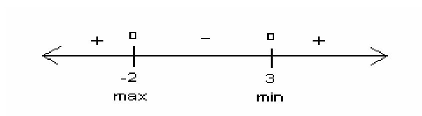
$y = 2x^3 - 3x^2 - 36x + 2$ *Find everything about this function*

$$\frac{dy}{dx} = 6x^2 - 6x - 36$$

$0 = 6(x^2 - x - 6)$ *1st derivative finds max, min, increasing,*

$0 = 6(x-3)(x-2)$

$x = 3, -2$



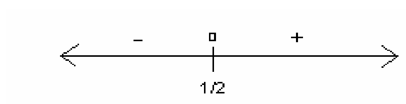
decreasing

max	**min**
(-2, 46)	(3, -79)
increasing	**decreasing**
$(-\infty$, -2] [3, $\infty)$	(-2, 3)

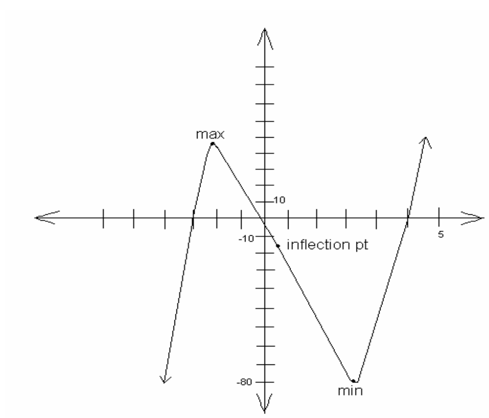
$$\frac{d^2y}{dx^2} = 12x - 6$$

$0 = 6(2x - 1)$ *2nd derivative finds concavity and inflection points*

$$x = \frac{1}{2}$$



inflection pt	
(½ , -16 ½)	
concave up	**concave down**
$(-\infty$, ½)	(1/2, $\infty)$



Miscellaneous

Newton's Method

Newton's Method is used to approximate a zero of a function

$$c - \frac{f(c)}{f'(c)} \qquad \textit{where } c \textit{ is the } 1^{st} \textit{ approximation}$$

Example:

If Newton's Method is used to approximate the real root of $x^3 + x - 1 = 0$, then a first approximation of $x_1 = 1$ would lead to a *third* approximation of x_3:

$$f(x) = x^3 + x - 1$$
$$f'(x) = 3x^2 + 1$$

$$1 - \frac{f(1)}{f'(1)} = \frac{3}{4} \quad or \quad .750 = x_2$$

$$\frac{3}{4} - \frac{f(\frac{3}{4})}{f'(\frac{3}{4})} = \frac{59}{86} \; or \; .686 = x_3$$

Separating Variables

Used when you are given the derivative and you need to take the integral. We separate variables when the derivative is a mixture of variables

Example:

If $\frac{dy}{dx} = 9y^4$ and if y = 1 when x = 0, what is the value of y when x = $\frac{1}{3}$?

$$\frac{dy}{dx} = 9y^4 \Rightarrow \frac{dy}{y^4} = 9\,dx$$

$$\int \frac{dy}{y^4} = \int 9\,dx \Rightarrow \frac{y^{-3}}{-3} = 9x + C$$

Continuity/Differentiable Problems

f(*x*) is continuous if and only if both halves of the function have the same answer at the breaking point.
f(*x*) is differentiable if and only if the derivative of both halves of the function have the same answer at the breaking point

Example:

$$f(x) = \begin{cases} \Rightarrow x^2, x \le 3 \\ \Rightarrow 6x - 9, x > 3 \end{cases} \qquad f'(x) = \begin{cases} \Rightarrow 2x = 6(\textit{plug in } 3) \\ \Rightarrow 6 = 6 \end{cases}$$

- At 3, both halves = 9, therefore, $f(x)$ is continuous
- At 3, both halves of the derivative = 6, therefore, $f(x)$ is differentiable

Useful Information

- We designate position as $x(t)$ or $s(t)$
- The derivative of position $x'(t)$ is $v(t)$, or velocity
- The derivative of velocity, $v'(t)$, equals acceleration, $a(t)$.
- We often talk about position, velocity, and acceleration when we're discussing particles moving along the x-axis.
- A particle is at rest when $v(t) = 0$.
- A particle is moving to the right when $v(t) > 0$ and to the left when $v(t) < 0$
- To find the average velocity of a particle: $\frac{1}{b-a}\int_a^b v(t)dt$

Average Value

Use this formula when asked to find the average of something

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Mean Value Theorem

NOT the same average value.

According to the Mean value Theorem, there is a number, *c*, between *a* and *b*, such that the slope of the tangent line at *c* is the same as the slope between the points (*a*, *f*(*a*)) and (*b*, *f*(*b*)).

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Growth Formulas

Double Life Formula: $y = y_0(2)^{t/d}$
Half Life Formula: $y = y_0(1/2)^{t/h}$
Growth Formula: $y = y_0 e^{kt}$

y = ending amount y_0 = initial amount t = time
k = growth constant d = double life time h = half life time

Useful Trig. Stuff

Double Angle Formulas:

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

Identities:

$\sin^2 x + \cos^2 x = 1$ $\qquad \frac{1}{\cos\theta} = \sec\theta$ $\qquad \frac{1}{\sin\theta} = \csc\theta$

$1 + \tan^2 x = \sec^2 x$ $\qquad \frac{\sin\theta}{\cos\theta} = \tan\theta$ $\qquad \frac{\cos\theta}{\sin\theta} = \cot\theta$

$1 + \cot^2 x = \csc^2 x$

Integration Properties

Area

$\int_a^b [f(x) - g(x)]dx$ $\quad$ $f(x)$ is the equation on top

Volume

$f(x)$ always denotes the equation on top

About the x-axis:

$\pi\int_a^b [f(x)]^2 dx$

$\pi\int_a^b [(f(x))^2 - (g(x))^2]dx$

About the y-axis:

$2\pi\int_a^b x[f(x)]dx$

$2\pi\int_a^b x[f(x) - g(x)]dx$

about line y = -1

$\pi\int_a^b [f(x) + 1]^2 dx$

about the line x = -1

$2\pi\int_a^b (x+1)[f(x)]dx$

Examples:

$f(x) = x^2$ [0,2]

x-axis:

$$\pi\int_0^2 (x^2)^2\,dx = \pi\int_0^2 x^4\,dx$$

y-axis:

$$2\pi\int_0^2 x[x^2]dx = 2\pi\int_0^2 x^3\,dx$$

about y = -1

In this formula <u>*f*(*x*) or *y*</u> is the radius of the shaded region. When we rotate about the line y = -1, we have to increase the radius by 1. That is why we add 1 to the radius

$$\pi\int_0^2 [x^2+1]^2\,dx = \pi\int_0^2 (x^4+2x^2+1)dx$$

about x = -1

In this formula, <u>*x*</u> is the radius of the shaded region. When we rotate about the line x = -1, we have the increased radius by 1.

$$2\pi\int_0^2 (x+1)[x^2]dx = 2\pi\int_0^2 (x^3+x^2)dx$$

<u>Trapeziodal Rule</u>

Used to <u>approximate</u> area under a curve using trapezoids.

Area $\approx \dfrac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\dots.+2f(x_{n-1})+f(x_n)\right]$

where n *is the number of subdivisions*

Example:

f(*x*) = x^2+1. Approximate the area under the curve from [0,2] using trapezoidal rule with 4 subdivisions

$$a = 0$$
$$b = 2$$
$$n = 4$$
$$A = \frac{2-0}{8}\left[f(0)+2f(.5)+2f(1)+2f(1.5)+f(2)\right]$$
$$= \frac{1}{4}\left[1+2(5/4)+2(2)+2(13/4)+5\right]$$
$$= \frac{1}{4}\left[(76/4)\right] = \frac{76}{16} = 4.750$$

<u>Riemann Sums</u>

Used to <u>approximate</u> area under the curve using rectangles.

a) Inscribed rectangles: all of the rectangles are below the curve

Example: $f(x) = x^2 + 1$ from [0,2] using 4 subdivisions
(Find the area of each rectangle and add together)

I=.5(1) II=.5(5/4) III=.5(2) IV=.5(13/4)
Total Area = 3.750

b) Circumscribed Rectangles: all rectangles reach above the curve

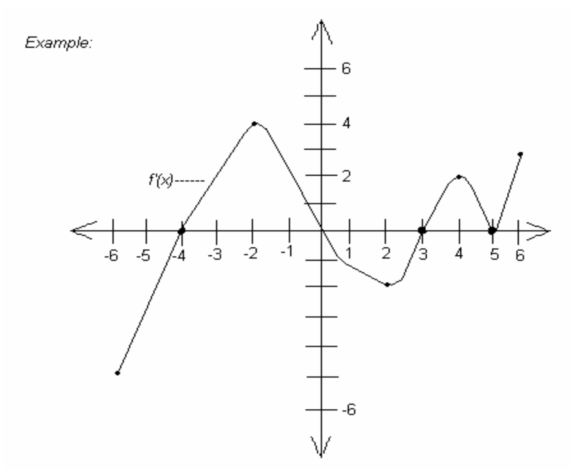
Example: $f(x) = x^2 + 1$ from [0,2] using 4 subdivisions

I=.5(5/4) II=.5(2) III=.5(13/4) IV=.5(5)
Total Area = 5.750

<u>Reading a Graph</u>

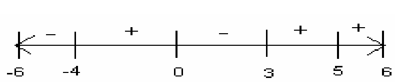
<u>When Given the Graph of f'(x)</u>

Make a number line because you are more familiar with number line.



This is the graph of $f'(x)$.
Make a number line.

- Where $f'(x) = 0$ (x-int) is where there are possible max and mins.
- Signs are based on if the graph is above or below the x-axis (determines increasing and decreasing)

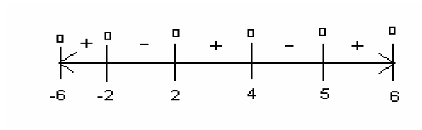


<u>min</u>	<u>max</u>
x = -4, 3	x = 0
<u>increasing</u>	<u>decreasing</u>
(-4,0) (3,6]	[-6,-4] (0,3)

- cont'd on next page -

To read the *f'*(*x*) and figure out inflection points and concavity, you read *f'*(*x*) the same way you look at *f*(*x*) (the original equation) to figure out max, min, increasing and decreasing.

For the graph on the previous page:



Signs are determined by if *f'*(*x*) is increasing (+) and decreasing (-)

<u>inflection pt</u>
x = -2, 2, 4, 5

<u>concave up</u>
(-6,-2) (2,4) (5,6)

<u>concave down</u>
(-2,2) (4,5)

Original Source:
http://www.geocities.com/Area51/Stargate/5847/index.html