Calculus with Algebra and Trigonometry II
Lecture 2
Applied optimization
or calculus word problems

Jan 22, 2015

Applied Optimization Strategy

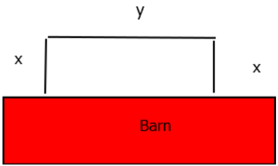
In order to solve these types of problem follow the steps below

- Read the problem carefully and identify the quantity to be maximized or minimized.
- Make a sketch of the situation and label the diagram with the variable(s) you are going to use
- Now write a formula for the quantity to be maximized/minimized
- If you have more than one independent variable, use the constraints to reduce the function to be maximized/minimized to a function of a single variable
- Use the first derivative to find the critical points - they are candidates for maxima and minima
- Use either the second derivative or the the sign of the first derivative near the critical point to determine whether it a maximum or a minimum
- Make sure you answer the question that was asked

Example 1

A farmer has 180 ft of fencing and wants to build a rectangular pen using his barn as one side. What is the maximum area he can enclose?

The situation is illustrated below



The function we want to maximize is the area

$$A = xy$$

This has two variables but we can use the fact that the farmer has 180 ft of fencing

$$2x + y = 180 \quad \Rightarrow \quad y = 180 - 2x$$

We can use this to eliminate y from the expression for the area

$$A(x) = x(180 - 2x) = 180x - 2x^2$$

Physically x must be greater than zero and less than 90 ft. The derivative is

$$A'(x) = 180 - 4x$$

The critical points are given by

$$A'(x) = 0 \quad \Rightarrow \quad 180 - 4x = 0 \quad \Rightarrow \quad x = 45\,\text{ft}$$

To check whether it is a maximum or a minimum, calculate the second derivative

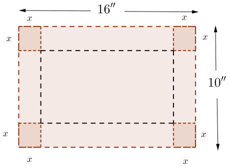
$$A''(x) = -4 < 0$$

so the function is concave and the critical point is a maximum. The max area is given by

$$A(45) = 45(180 - 2(45)) = 4050\,\text{ft}^2$$

Example 2

You make a box out of a rectangle of cardboard by cutting four equal squares out of the corners and folding up the sides. Given a sheet $16'' \times 10''$ what is the maximum volume the box can have?



If the length of the sides of the cut out squares is x, then the volume will be given by

$$V(x) = (16 - 2x)(10 - 2x)x = 4x^3 - 52x^2 + 160x$$

Physically the allowable values of x satisfy $0 \leq x \leq 5$. The derivative of $V(x)$ is

$$V'(x) = 12x^2 - 104x + 160$$

The critical points satisfy

$$12x^2 - 104x + 160 = 0$$

Using the quadratic formula gives $x = 2, \frac{20}{3}$. But $\frac{20}{3} > 5$ and so is not a physically reasonable answer. To check the point $x = 2$, calculate the second derivative.

$$V''(x) = 24x - 104 \quad \Rightarrow \quad V''(2) = -56 < 0$$

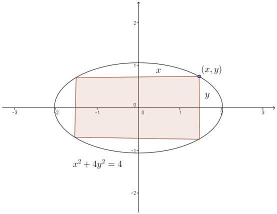
So the function is concave at $x = 2$ and so $x = 2$ is a maximum. The maximum value is

$$V(2) = 2(16 - 4)(10 - 4) = 144\,\text{in}^3$$

Example 3

What is the rectangle with the largest area that be inscribed in the ellipse

$$x^2 + 4y^2 = 4$$



The area we would like to maximize is given by

$$A = 4xy$$

The point (x, y) is on the ellipse so

$$x^2 + 4y^2 = 4 \quad \Rightarrow \quad x = 2\sqrt{1 - y^2}$$

Using this to eliminate x (for a change) gives

$$A(y) = 8y\sqrt{1-y^2}$$

Physically the domain for y is $0 \le y \le 1$. The derivative of $A(y)$ is

$$A'(y) = 8\sqrt{1-y^2} + 8y\left(\frac{1}{2}(1-y^2)^{-1/2}(-2y)\right) = \frac{8(1-2y^2)}{\sqrt{1-y^2}}$$

The critical points are

$$A'(y) = 0 \quad \Rightarrow \quad 1 - 2y^2 = 0 \quad \Rightarrow \quad y = \frac{1}{\sqrt{2}}$$

Now

$$y < \frac{1}{\sqrt{2}} \quad \Rightarrow \quad A'(y) > 0 \qquad y > \frac{1}{\sqrt{2}} \quad \Rightarrow \quad A'(y) < 0$$

So the function is increasing then decreasing as you move through $y = \frac{1}{\sqrt{2}}$ as thus it is a maximum. The maximum area is then

$$A\left(\frac{1}{\sqrt{2}}\right) = 8\left(\frac{1}{\sqrt{2}}\right)\sqrt{1 - \frac{1}{2}} = 4$$

Example 4

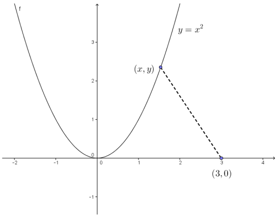
Find the minimum distance from the point (3,0) to the parabola $y = x^2$.

The distance from (3,0) to any point (x, y) is

$$d = \sqrt{(x-3)^2 + y^2}$$

To simplify the algebra we are going to minimize d^2 instead

$$D = d^2 = (x-3)^2 + y^2$$



The point (x, y) is on the parabola $y = x^2$ thus we can use this to eliminate y.

$$D(x) = (x-3)^2 + x^4 = x^4 + x^2 - 6x + 9$$

The domain for x will be $0 \leq x \leq 3$. The derivative is

$$D'(x) = 4x^3 + 2x - 6 = 2(x-1)(2x^2 + 2x + 3)$$

The critical points are given by

$$D'(x) = 2(x-1)(2x^2 + 2x + 3) = 0 \quad \Rightarrow \quad x = 1$$

To show it is a minimum use the second derivative

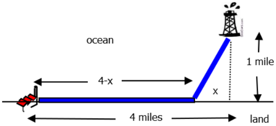
$$D''(x) = 12x^2 + 2 \quad \Rightarrow \quad D''(1) = 14 > 0$$

So the function is convex at $x = 1$ and so it is indeed a minimum

Example 5

It is desired to build a pipeline from an at sea oil well to refinery on the shore. The oil weel is 1 mile offshore and the refinery is 4 miles along the coastline. Building a pipe costs $ 500,000 per mile under water and $ 300,000 per mile under land. Find the cost of the cheapest possible pipeline.

The situation is pictured below



The cost of the pipeline will be

$$C(x) = 300000(4-x) + 500000\sqrt{x^2+1}$$

and the domain for x is clearly $0 \leq x \leq 4$

The derivative is

$$C'(x) = -300000 + 500000\frac{x}{\sqrt{x^2+1}}$$

The critical points are given by

$$-300000 + 500000\frac{x}{\sqrt{x^2+1}} = 0 \quad \Rightarrow \quad 3\sqrt{x^2+1} = 5x$$

$$9(x^2+1) = 25x^2 \quad \Rightarrow \quad 9 = 16x^2 \quad \Rightarrow \quad x = \frac{3}{4}$$

The second derivative is

$$C''(x) = \frac{500000}{(x^2+1)^{3/2}} > 0$$

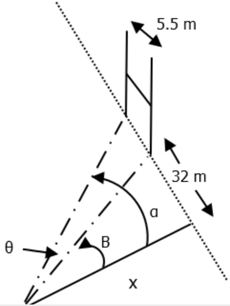
The function is convex and $x = \frac{3}{4}$ gives a minimum. The minimum cost is then

$$C\left(\frac{3}{4}\right) = \$1600000$$

Example 6

In rugby when you score the scoring team gets to free kick at the goal. The ball must be kicked on a line from the place you scored perpendicular to the goal line. Given your team scored 32 m from the near post and the fact that the posts are 5.5 m apart what is best place to kick from to give you the largest target.

What we are trying to maximize is the angle between the two goalpost viewed from the point at which the kick is made (θ in the diagram below).



From the geometry we have

$$\tan\alpha = \frac{37.5}{x} \qquad \tan\beta = \frac{32}{x} \qquad \theta = \alpha - \beta$$

$$\theta(x) = \tan^{-1}\left(\frac{37.5}{x}\right) - \tan^{-1}\left(\frac{32}{x}\right)$$

The derivative is

$$\begin{aligned}\theta'(x) &= \frac{1}{1+(\frac{37.5}{x})^2}\left(-\frac{37.5}{x^2}\right) - \frac{1}{1+(\frac{32}{x})^2}\left(-\frac{32}{x^2}\right) \\ &= \frac{32}{x^2+32^2} - \frac{37.5}{x^2+37.5^2}\end{aligned}$$

Using common denominators and simplifying we can express the derivative as

$$\theta'(x) = \frac{5,5(1200 - x^2)}{(x^2 + 32^2)(x^2 + 37.5^2)}$$

The critical point is given by

$$1200 - x^2 = 0 \quad \Rightarrow \quad x = 20\sqrt{3}\,\mathrm{m}$$

To check whether it it is a maximum note

$$x < 20\sqrt{3} \quad \Rightarrow \quad \theta'(x) > 0$$

$$x > 20\sqrt{3} \quad \Rightarrow \quad \theta'(x) < 0$$

So it is a maximum